INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & MANAGEMENT

ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $3(x^2+y^2)-2xy=4z^2$

M.A.Gopalan [1]*, S.Vidhyalakshmi [2], A.Kavitha [3], D.Mary Madona [4]

[1,2] Professor, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu,

[3] Lecturer, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

[4] M.Phil. Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamil Nadu

mayilgopalan@gmail.com

ABSTRACT:

The ternary homogeneous quadratic equation given by $3(x^2+y^2)-2xy=4z^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented. Also, given a solution, formula for generating a sequence of solutions based on the given solution is presented.

KEYWORDS: Ternary quadratic , integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal number.

2010 Mathematics subject classification: 11D09.

NOTATIONS USED:

1. Polygonal number of rank 'n' with sides m

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

2. Stella octangular number of rank 'n'

$$SO_n=n(2n^2-1)$$

3. Pronic number of rank 'n'

$$Pr_n=n(n+1)$$

4. Octahedral number of rank 'n'

$$OH_n=\frac{1}{3}[n(2n^2+1)]$$

I. INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety[1-3].In particular, one may refer [4-24] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $3(x^2+y^2)-2xy=4z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solution is presented.

II. METHOD OF ANALYSIS

Consider the equation

$$3(x^2+y^2)-2xy=4z^2 \tag{1}$$

The substitution of the linear transformations

$$x=u+v\ ;\ y=u-v \tag{2}$$

in (1) gives $u^2+2v^2=z^2$ (3)

We present below different methods of solving(3) and thus obtain different choices of integer solutions of (1)

METHOD:1

Write (3) in the form of ratio as

$$\frac{2v}{z-u} = \frac{z-u}{v} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{4}$$

This is equivalent to the following two equation,

$$\alpha u - 2\beta v + \alpha z = 0$$
$$\beta u + \alpha v - \beta z = 0 \tag{5}$$

Applying the method of cross multiplication ,the above system of equations (5) is satisfied by

$$u = 2\beta^2 - \alpha^2, v = \alpha^2 + \beta^2$$

$$z = z(\alpha, \beta) = 2\beta^2 + \alpha^2 \tag{6}$$

Substituting the values of u and v in (2),we get

$$x = x(\alpha, \beta) = 2\beta^2 - \alpha^2 + 2\alpha\beta$$
$$y = y(\alpha, \beta) = 2\beta^2 - \alpha^2 - 2\alpha\beta \tag{7}$$

Thus (6) and (7) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

A few interesting properties obtained as follows:

- $x(1, a) - y(1,a) - 4t_{4,a} - 4Pr_a - 4t_{4,a} = 0$
- $x(a, 2a) - y(a,2a) - 4t_{6,n} \equiv 0 \pmod 4$
- $x(a, 2a^2 - 1) - y(a, 2a^2 - 1) - 4so_a \equiv 0 \pmod 4$
- $x(a,1) - y(a,1) - z(a,1) + pr_a \equiv -2 \pmod 5$
- $x(a, 2a) + y(a,2a) - 4t_{10,n} - t_{6,n} \equiv 0 \pmod{11}$

Remark:

In addition to (4), (3) may also be expressed in the form of ratios in three different ways that are presented below:

Way 1: $\frac{v}{z+u} = \frac{z-u}{2v} = \frac{\alpha}{\beta}$

Way 2: $\frac{2v}{z-u} = \frac{z+u}{v} = \frac{\alpha}{\beta}$

Way 3: $\frac{v}{z-v} = \frac{z+u}{2v} = \frac{\alpha}{\beta}$

Solving each of the above system of equations by following the procedure presented in Method 1, the corresponding integer solutions to (1) are found to be as given below:

Solution for way 1:

$$x(\alpha, \beta) = \beta^2 - 2\alpha^2 + 2\alpha\beta$$
$$y(\alpha, \beta) = \beta^2 - 2\alpha^2 - 2\alpha\beta$$
$$z(\alpha, \beta) = 2\alpha^2 + 2\beta^2$$

Solution for way 2:

$$x(\alpha,\beta) = -\alpha^2 + 2\beta^2 - 2\alpha\beta$$
$$y(\alpha,\beta) = -\alpha^2 - 2\beta^2 + 2\alpha\beta$$
$$z(\alpha,\beta) = -\alpha^2 - 2\beta^2$$

Solution for way 3:

$$x(\alpha,\beta) = \beta^2 - 2\alpha^2 - 2\alpha\beta$$
$$y(\alpha,\beta) = \beta^2 - 2\alpha^2 + 2\alpha\beta$$
$$z(\alpha,\beta) = -2\alpha^2 - \beta^2$$

METHOD 2:

(3) is written as

$$u^2 + 2v^2 = z^2 = z^2 * 1 \quad (8)$$

Assume $z = z(a,b) = a^2 + 2b^2$ (9)

where a, b are non –zero integers, write 1 as

$$1 = \frac{(1+i2\sqrt{2})(1-i\,2\sqrt{2})}{9} \quad (10)$$

Substituting (9) and(10) in (8) , it is written in the factorizable form as

$$(u+i\sqrt{2}v)(u-i\sqrt{2}v) = \frac{(1+i2\sqrt{2})}{3}\frac{(1-i2\sqrt{2})}{3}(a+i\sqrt{2}b)^2(a-i\sqrt{2}b)^2$$

Equating the positive & negative factors,we get

$$(u+i\sqrt{2}v) = \frac{(1+i2\sqrt{2})}{3}(a+i\sqrt{2}b)^2 \quad (11)$$

$$(u-i\sqrt{2}v) = \frac{(1-i2\sqrt{2})}{3}(a-i\sqrt{2}b)^2 \quad (12)$$

On equating real and imaginary parts in either(11) or (12), we get

$$u = \frac{(a^2 - 2b^2 - 8ab)}{3}$$

$$v = \frac{(2a^2 - 4b^2 + 2ab)}{3}$$

Substituting the values u and v in(2),we have

$$x = x(a,b) = (a^2 - 2b^2 - 2ab) \quad (13)$$

$$y = y(a,b) = \frac{(-a^2 + 2b^2 - 10ab)}{3} \quad (14)$$

As our interest centers on finding integers solutions, it is seen that x and y are integers for suitable choices of a and b .Replacing a by 3a and by 3b, the corresponding non zero distinct integer solutions to (1) are given by

$$x = x(a,b) = 9a^2 - 18b^2 - 18ab$$
$$y = y(\alpha,\beta) = -3a^2 + 6b^2 - 30ab$$
$$z = z(a,b) = 9a^2 + 18b^2$$

Properties:

A few interesting properties obtained as follows:

- $x(2a,3) - y(2a,3) - 12\,t_{10,n} \equiv -216 \pmod{108}$

- $2x(1,a) - 2z(1,a) - 12\, t_{8,n} \equiv 24 \pmod{36}$
- $2y(a+1, a+1) - z(a+1, a+1) + 56\, t_{8,n} \equiv -216 \pmod{81}$
- $y(1,a) - z(1,a) + 3t_{10,n} \equiv -12 \pmod{39}$
- $2y(1,a) + 2z(1,a) - 12t_{10,n} \equiv -12 \pmod{24}$

METHOD:3
In addition to (9), write 1 as

$$1 = \frac{(7 + i\,6\sqrt{2})(7 - i6\sqrt{2})}{121} \tag{15}$$

Substituting (9) and(15) in (8) , it is written in the factorizable form as

$$(u + i\sqrt{2}v)(u - i\sqrt{2}v) = \frac{(7 + i6\sqrt{2})(7 - i6\sqrt{2})}{121}(a + i\sqrt{2}b)^2 (a - i\sqrt{2}b)^2$$

Equating the positive & negative factors ,we get

$$(u + i\sqrt{2}v) = \frac{(7 + i6\sqrt{2})}{11}(a + i\sqrt{2}b)^2 \tag{16}$$

$$(u - i\sqrt{2}v) = \frac{(7 - i6\sqrt{2})}{11}(a - i\sqrt{2}b)^2 \tag{17}$$

On equating real and imaginary parts in either(15) or (16) , we get

$$u = \frac{(7a^2 - 14b^2 - 24ab)}{11}$$

$$v = \frac{6a^2 - 12b^2 + 14ab}{11}$$

Substituting the values u and v in(2),we have

$$x = x(a,b) = \frac{(13a^2 - 26b^2 - 10ab)}{11} \tag{18}$$

$$y = y(a,b) = \frac{(a^2 - 2b^2 - 38ab)}{11} \tag{19}$$

As our interest centers on finding integers solutions, it is seen that x and y are integers for suitable choices of a and b .Replacing a by 11a and by 11b, the corresponding non zero distinct integer solution to (1) are given by

$$x = x(a,b) = 143a^2 - 286b^2 - 110ab$$
$$y = y(\alpha,\beta) = 11a^2 - 22b^2 - 418ab$$
$$z = z(a,b) = 121a^2 + 242b^2$$

Properties:
A few interesting properties obtained as follows:

- $x(a,1) - z(a,1) - 11t_{6,n} \equiv -528 \pmod{99}$
- $x(2a, a) + z(2a,a) - 1056t_{4,n} - 22t_{6,n} - 55\, t_{10,n} \equiv 0 \pmod{187}$
- $x(a,1) - y(a,1) - 33\, t_{10,n} \equiv -264 \pmod{407}$
- $x(2a, a) + y(2a,a) - 308t_{6,n} - 1364t_{4,n} \equiv 0 \pmod{308}$
- $y(1,a) + x(1, a) + 77t_{10,n} \equiv 154 \pmod{759}$

METHOD:4

Write (3) as

$$z^2 - 2v^2 = u^2 = u^2 * 1 \tag{20}$$

Write 1 as

$$1 = (3 + 2\sqrt{2})(3 - 2\sqrt{2}) \tag{21}$$

Assume u = u (a, b)= $a^2 - 2b^2$ (22)

where a ,b are non –zero integers, Using (20) and (21) in (22) ,it is written in the factorizable form as

$$(z + \sqrt{2}v)(z - \sqrt{2}v) = (3 + 2\sqrt{2})(3 - 2\sqrt{2})(a + \sqrt{2}b)^2 (a - \sqrt{2}b)^2$$

Equating the positive & negative factors ,we get

$$(z + \sqrt{2}v) = (3 + 2\sqrt{2})(a + \sqrt{2}b)^2 \tag{23}$$

$$(z - \sqrt{2}v) = (3 - 2\sqrt{2})(a - \sqrt{2}b)^2 \tag{24}$$

On equating rational and irrational parts in either(23) or (24), we get

$$z = 3a^2 + 6b^2 + 8ab$$

$$v = 2a^2 + 4b^2 + 6ab \tag{25}$$

Substituting the values u and v in (2),we have

$$x = x(a,b) = 3a^2 + 2b^2 + 6ab$$

$$y = y(a,b) = -a^2 - 6b^2 - 6ab \tag{26}$$

Note that (25) and (26) represent the non –zero distinct interger solutions to (1)

Properties:

A few interesting properties obtained as follows:

- $x(a, a) + y(a,a) - t_{6,n} + t_{10,n} \equiv 0 \pmod 2$
- $x(2a, a) - y(2a,a) - 4t_{10,n} - 4t_{6,n} - 24\, t_{4,n} \equiv 0 \pmod{16}$
- $x(a, a+1) - z(a,a+1) + 2\, t_{8,n} \equiv -4 \pmod{14}$
- $x(a+1, a) + y(a+1, a) + t_{6,n} \equiv -2 \pmod 3$
- $z(2a, a) + y(2a,a) + x(2a, a) - 5t_{4,n} - 4t_{6,n} - 4t_{10,n} \equiv 0 \pmod{16}$

III. Remarkable observations

If the non zero integer triple ($x_{0,}\, y_{0,}\, z_0$) is any solution of(1) ,then a general formula for generating a sequence of solutions based on the given solution is illustrated below :

Let ($x_{0,}\, y_{0,}\, z_0$) be any given solution of (1) and let the second solution of (1) be,

$$\begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 \qquad h \neq 0 \\ z_1 &= h - z_0 \end{aligned} \tag{27}$$

Substituting (27S) in (1) & performing a few calculations, we have

$h = 6x_0 + 8z_0$ and then

$$x_1 = 7x_0 + 8z_0$$

$$z_1 = 6x_0 + 7z_0$$

This is written in the form of matrix as $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ (28)

where

$$M = \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}$$

From the second solution, we get the corresponding third solution ($x_{2,}\, y_{2,}\, z_2)$ as given below.

$$x_2 = 97x_0 + 112z_0$$

$$y_2 = y_0$$

$$z_2 = 90x_0 + 104z_0$$

Repeating the above process, the general solution (x_n, y_n) to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{29}$$

To find M^n, the Eigen values of M are $\alpha = 7 + 4\sqrt{3}, \beta = 7 - 4\sqrt{3}$.
It is well-known that

$$M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$$

Using the above formula, we have

$$M^n = \frac{1}{8\sqrt{3}} \begin{bmatrix} 4\sqrt{3}A^n & 8B^n \\ 6B^n & 4\sqrt{3}A^n \end{bmatrix}$$

Where $A^n = (\alpha^n + \beta^n)$

$B^n = (\alpha^n - \beta^n)$

Hence we get the n^{th} solution to be

$$x_n = \frac{1}{2}\left[(4\sqrt{3}A_n x_0 + 8B_n z_0\right]$$

$$y_n = y_0$$

$$Z_n = \frac{1}{8\sqrt{3}}\left[(6B_n x_0 + 4\sqrt{3}A_n z_0\right]$$

IV. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by

$$3(x^2 + y^2) - 2xy = 4z^2$$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

V. Acknowledgement:

The financial support from the UGC, New Delhi (F-MRP-5122/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

References

[1] Dickson. I.E" History of Theory of Numbers", Volume 2. Diophantine analysis, New York, Dover, 2005.
[2] Model .L.J.," Diophantine Equations" Academic Press, New York, 1969.
[3] Carmichael.R.D"The Theory of numbers and Diophantine Analysis", New York, Dover, 1959.
[4] M.A.Gopalan, Geetha D and Lattice points on the Hyberboloid of two sheets $x^2+6xy+y^2+6x-2y+5=z^2+4$, Impact J Sci Tech 2010;4:23-32
[5] M.A.Gopalan and Vidhyalakshmi S, Kavitha A . Integral points on the Homogeneous cone $z^2=2x^2-7y^2$,The Diophantus J M ath 2012;1(2):127-136.
[6] M.A.Gopalan, Vidhyalakshmi S, Sumati G ,Lattice points on the Hyberboloid one sheet $z^2=2x^2+3y^2-4$.Diophantus J Math 2012;1(2):109-115
[7] M.A.Gopalan and Vidhyalakshmi S ,lakshmi k. Intergral points on the Hyberboloid of two sheets $3y^2=7x^2-z^2+21$ Diophantus J Math 2012;1(2):99-107.
[8] Gopalan MA, Vidhyalakshmi S, Mallika S. Observation on Hyperboloid of one sheet $x^2+2y^2-z^2=2$ Bessel J of Math 2012; 2(3):221-226.
[9] Gopalan MA, Vidhyalakshmi S, Usha Rani TR. Mallika S. $6z^2+3y^2-2x^2=0$ Integral points on the Homogeneous cone. Impact J Sci Tech 2012;6(1):7-13.

[10]. Gopalan MA, Vidhyalakshmi S, Sumathi G. Lattice points on the Elliptic parabol$9x^2+4y^2=z$ Advances in Theoretical and Applied Mathematics2012; 7(4):379-385.

[11]. Gopalan MA, Vidhyalakshmi S, Usha Rani TR. Integralpoints on the non-homogeneous cone $2z^2+4xy+8x-4z=0$. Global Journal ofMathematics and Mathematical Sciences 2012;2(1):61-67.

[12]. Gopalan MA, Vidhyalakshmi S, Lakshmi K. Lattice points on the elliptic paraboloid$16y^2+9z^2=4x$ Bessel J of Math 2013; 3(2):137-145.

[13]. Gopalan MA, Vidhyalakshmi S, UmaRani J, Integralpoints on the homogeneous cone $x^2+4y^2=37z^2$ Cayley J of Math, 2013, 2(2):101-107.

[14]. Gopalan MA, Vidhyalakshmi S, Kavitha A.Observations on the Hyperboloid of two Sheets $7x^2-3y^2=z^2+z(y-x)+4$. InternationalJournal of Latest Research in Science and technology 2013; 2(2):84-86.

[15]. Gopalan MA, Sivagami B. Integral points on the homogeneous cone $z^2=3x^2+6y^2$ IOSR Journal ofMathematics 2013; 8(4):24-29.

[16]. Gopalan MA, Geetha V. Lattice points on the Homogeneous cone $z^2=2x^2+8y^2-6xy$. Indianjournal of Science 2013; 2:93-96.

[17]. Gopalan MA, Vidhyalakshmi S, Maheswari D. Integral points on the homogeneous cone $2x^2+3y^2=35z^2$ Indian Journal of Science 2014; 7:6-10.

[18]. Gopalan. MA, Vidhyalakshmi S, Umarani J. On theTernary Quadratic Diophantine equation $6(x^2+y^2)-8xy=21z^2$ Sch J Eng Tech 2014;2(2A):108-112.

[19]. Meena K, Vidhyalakshmi S, Gopalan MA, Priya IK. Integral points on the cone $3(x^2+y^2)-5xy=47z^2$ Bulletin of Mathematics and Statistic Reseach 2014; 2(1):65-70.

[20]. Gopalan MA, Vidhyalakshmi S, Nivetha S. On Ternary Quadratics equation $4(x^2+y^2)-7xy=31z^2$ Diophantus J Math 2014; 3(1):1-7.

[21]. Meena K, Vidhtalakshmi S, Gopalan MA, Thangam SA. Integer Solutions on the Homogeneous cone $4x^2+3y^2=28z^2$ Bulletin of Mathematics and statistics Research 2014; 2(1):47-53.

[22]. Shanthi J, Gopalan MA, Vidhyalakshmi. Lattice pointson the Homogeneous cone $8(x^2+y^2)-15xy=56z^2$ Sch Journal of PhyMath Stat 2014; 1(1):29-32.

[23]. Gopalan MA, Vidhyalakshmi S, Mallika S. On ternary quadratic Diophantine equation $8(x^2+y^2)15xy=80z^2$, BOMSR, 2014; 2(4):429-433.

[24] .Gopalan MA, Meena k, Vidhyalakshmi S, Thiruniraiselvi N. On Ternary Quadratics equation $x^2+9y^2=50z^2$. Diophantus J Math 2015; 1(2):51-53.